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THE WHITEHEAD TRANSFER HOMOMORPHIM FOR ORIENTED S^1-BUNDLES

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Consider the following problem.

<u>Question</u>. Given an oriented circle fiber bundle $S^1 \to E \to B$ and a homotopy equivalence $\varphi: B' \to B$, where E, B, and B' all are finite complexes; is the pullback homotopy equivalence $\tilde{\varphi}: \varphi^*(E) \to E$ always a simple homotopy equivalence? If not, what can one say about the torsion of $\tilde{\varphi}$?

This question arose naturally from the theorem that $Id_{S^1} \times \varphi$ is a simple homotopy equivalence whenever φ is a homotopy equivalence of finite complexes [1, Corollary 1.4]. If one works with unoriented S^1-bundles, it is easy to find examples where $\tau(\tilde{\varphi}) \neq 1$, using, e.g.,[1, Corollary C]. The main new result is that examples with $\tau(\tilde{\varphi}) \neq 1$ also occur for oriented S^1-bundles. Proving the existence of such examples is surprisingly complicated, and provides good test of the current machinery for describing Whitehead groups of finite groups.

The above problem was studied by Munkholm and Pedersen in [2]. They showed that there is a well defined "transfer" map

$$f^{\#}: Wh(\pi_1(B)) \to Wh(\pi_1(E))$$

for any bundle $S^1 \to E \to B$ (not necessarily oriented), depending only on the fundamental group sequence $\mathbb{Z} \to \pi_1 E \to \pi_1 B$, and such that for any homotopy equivalence $\varphi\colon B' \to B$,

$$\tau\left(\tilde{\varphi}\colon \varphi^*(E) \to E\right) = f^\#(\tau(\varphi)).$$

Conversely, any exact sequence $\mathbb{Z} \xrightarrow{i} \tilde{G} \xrightarrow{\beta} G \to 1$ of finitely presented groups with $i(\mathbb{Z}) \subseteq Z(\tilde{G})$ can be realized as the fundamental group sequence of some oriented bundle $S^1 \to E \to B$ of finite complexes (see [3, Proposition 11.4 and the following remark]). Such a sequence is described by the pair $(\beta, i(1))$. In other words, given any surjection $\beta\colon \tilde{G} \to G$ of groups such that $\mathrm{Ker}(\beta)$ is cyclic and central in $\tilde{G}$, and given any generator $z \in \mathrm{Ker}(\beta)$, there is an induced homomorphism

$$\beta_z^\#\colon \mathrm{Wh}(G) \to \mathrm{Wh}(\tilde{G})$$

such that $\beta_z^\# = f^\#$ for any oriented bundle $S^1 \to E \to B$ realizing β and z. This map was described algebraically in [2], and the authors there also listed many cases when it is trivial. For example, they showed that $\beta_z^\# = 1$ if $\tilde{G}$ is infinite and G is finite, or if $\tilde{G}$ is finite abelian, or more generally if $\tilde{G}$ is finite and $\mathrm{Ker}(\beta) \cap [\tilde{G},\tilde{G}] = 1$.

Since $\mathrm{Wh}(G)$ is so poorly understood for infinite G, the natural place to look for examples of $\beta_z^\# \neq 1$ is when $\tilde{G}$ and G are both finite. By [2, Proposition 6.2],

$$\mathrm{Im}(\beta_z^\#) \subseteq Cl_1(\mathbb{Z}\tilde{G}) = \mathrm{Ker}[K_1(\mathbb{Z}\tilde{G}) \to K_1(\mathbb{Q}\tilde{G}) \times \prod_p K_1(\hat{\mathbb{Z}}_p\tilde{G})]$$

in this case. This is the first indication of difficulties in detecting elements in $\mathrm{Im}(\beta_z^\#)$: Concrete calculations in $Cl_1(\mathbb{Z}\tilde{G})$ involve almost automatically calculations with $K_2(\hat{\mathbb{Q}}_p\tilde{G})$ and $K_2(\hat{\mathbb{Z}}_p\tilde{G})$ for primes $p \mid |G|$. The second indication is that $\beta_z^\#(\mathrm{Wh}(G))/\beta_z^\#(SK_1(\mathbb{Z}G))$ has exponent at most 2 - in fact, we show that $\mathrm{Im}(\beta^{\#}_z) = \beta^{\#}_z(SK_1(\mathbb{Z}G))$ if G has normal 2-Sylow subgroup, and we know of no example where the groups differ. Finally, $Cl_1(\mathbb{Z}G) \subseteq \mathrm{Ker}(\beta^{\#}_z)$, and

$$SK_1(\mathbb{Z}G)/Cl_1(\mathbb{Z}G) \cong \prod_p SK_1(\hat{\mathbb{Z}}_pG)$$

is non-zero only when G is fairly complicated - sufficiently complicated that describing $Cl_1(\mathbb{Z}G)$ becomes (at best) a rather messy combinatorial problem.

In [4], examples with $\beta^{\#}_z \neq 1$ are constructed, showing, in fact, that p-torsion can occur in $\mathrm{Im}(\beta^{\#}_z)$ for any prime p. When p is odd, one can take

$$\tilde{G} = \langle a_1, a_2, a_3, a_4, z \mid a_i^p = 1 = [a_i, [a_j, a_k]] = [a_i, z] \text{ (all } i, j, k), \\ z^p = [a_1, a_2][a_3, a_4] \rangle ,$$

$G = \tilde{G}/\langle z\rangle$, and $\beta: \tilde{G} \to G$ is the projection.

The $\beta^{\#}_z$ are then further analyzed, in an attempt to explain why such examples need be so complicated. The main general results, for surjections $\beta: \tilde{G} \to G$ of finite groups with $\mathrm{Ker}(\beta)$ central and cyclic, are:

(a) $\beta^{\#} = \beta^{\#}_z$ is independent of the choice of generator $z \in \mathrm{Ker}(\beta)$ (this is not at all obvious from the geometry).

(b) for any prime p, if $K = \{g \in \mathrm{Ker}(\beta) : p \nmid |g|\}$, then

$$\mathrm{Ker}[\beta^{\#}_{(p)} : Wh(G) \to Cl_1(\mathbb{Z}G)_{(p)}] = \mathrm{Ker}(\beta/K)^{\#} : Wh(G) \to Cl_1(\mathbb{Z}[\tilde{G}/K])]$$

(see [4, Theorem 4.10] for more detail). The problem of describing $\beta^{\#}$ is thus reduced to the case where $\mathrm{Ker}(\beta)$ is a p-group.

(c) if $\mathrm{Ker}(\beta)$ is a p-group for some p, then $\beta^{\#}$ factors as a composite

$$\beta^{\#}: Wh(G) \xrightarrow{\Omega_\beta} W_\beta / W'_\beta \xrightarrow{\Theta_\beta} Cl_1(\widetilde{\mathbb{Z}G}),$$

where $W'_\beta \subseteq W_\beta$ are defined explicitly in [4, Proposition 4.8].

The problem of describing $\beta^{\#}$ is thus split into two independent problems: one involving $Wh(G)$, and the other involving $Cl_1(\widetilde{\mathbb{Z}G})$. For more detail on this, see Theorem 4.9 in [4] and the following discussion. Some general consequences of the above decomposition are listed in [4, Theorem 4.11].

1. K. Kwun and R. Szczarba, Product and sum theorems for Whitehead torsion, Ann. Math. 82 (1965), 183-190.

2. H. Munkholm and E. Pederson, Whitehead transfers for S^1-bundles, an algebraic description, Comment. Math. Helv. 56 (1981), 404-430.

3. C. T. C. Wall, Surgery on compact manifolds, Academic Press (1970).

4. R. Oliver, The Whitehead transfer homomorphism for oriented S^1-bundles, Math. Scand. (to appear).